Exercise Sheet 8

You will need the following results:

Mean Value Principle: Let u be harmonic in some open $\Omega \subset \mathbb{C}$ nad let $\overline{D}_r(a) \subset W$, Then:

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})dt.$$

Harmonic Conjugate Let u be a harmonic function in an open set $W \subset \mathbb{C}$. A harmonic function v in W is called the harmonic conjugate of u, if $u + iv$ is holomorphic in W.

Schwartz' Lemma: Let f be a holomorphic function on the unit disc, $D_1(0)$, and assume that $f(0) = 0$ and $|f(z)| \leq 1$, for every $z \in D_1(0)$. Then for every $z \in D_1(0)$, we have that $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Furthermore if there exists a point $0 \neq z_0 \in D_1(0)$, such that $|f(z_0)| = |z_0|$, then there exists a constant c, with $|c| = 1$, such that $f(z) = cz$.

Möbius Transforms For every two triplets of distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, there exists a unique Möbius transform, φ, satisfying $\varphi(z_j) = w_j$, $j = 1, 2, 3$.

1. Show that the following functions are harmonic in the plane and find the harmonic conjugates:
   - $u(x, y) = xy + 3x^2y - y^3$;
   - $u(x, y) = \sinh(x)\sin(y)$;
   - $u(x, y) = e^{x^2-y^2}\cos(2xy)$.
2. Prove the following facts about harmonic functions:
   - Let u, v be harmonic in some open set containing a domain D. If $u = v$ on the boundary of D, then $u = v$ in D.
   - Every partial derivative of a harmonic function is harmonic.
   - The zeroes of harmonic functions are never isolated.
3. Let $\varphi(z) = \frac{az+b}{cz+d}$ be a Möbius transform. Find an expression for the fixed points of φ. How many are there, if $\varphi(z) \neq z$, for some z.
4. - Denote $d_a(z) = az$, for $a \in \mathbb{C} \setminus \{0\}$ (this map is called dilation), $t_b(z) = z + b$, for $b \in \mathbb{C}$ (this map is called a translation) and $\iota(z) = \frac{1}{z}$ (inversion). Show that every Möbius transform is a composition of dilations, translations and inversions. Hint: first isolate those transform that fix ∞ and show the claim, then take a transform φ, such that $\varphi(\infty)$ is finite, write $\varphi(z) = \frac{az+b}{z+d}$ (why can you do that?) and deduce the claim.
   - Using the above result conclude that every Möbius transform maps circles and lines to circles and lines.
5. Let φ be a Möbius transform. Denote by $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. If $\varphi(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$, show that $\varphi(z) = \frac{az+b}{cz+d}$, such that $a, b, c, d \in \mathbb{R}$.
6. (*) Given a function, g, continuous on the imaginary axis, $i\hat{\mathbb{R}}$, show that there exists a function, u, continuous on the right half plane and the imaginary axis and harmonic on the right half plane, such that u agrees with g on the imaginary axis. Hint: Use a Möbius transform.
7. Denote $S = \{z \in \mathbb{C} \mid |z| = 1\}$, the unit circle. Find all the Möbius transforms, φ, such that $\varphi S = S$.
8. Find a Möbius transform that sends D_1 to D_2 or show that there exists no such transform, where:

- $D_1 = \{z \in \mathbb{C} \mid |z| < 1 \text{ \& } \operatorname{Im}(z) > 0\}$ and $D_2 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \text{ \& } \operatorname{Re}(z) > 0\}$;
- $D_1 = D_1(1+i)$ and $D_2 = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > \operatorname{Re}(z)\}$;
- $D_1 = D_1(1+i) \cup D_1(-1+i)$ and $D_2 = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$;
- $D_1 = D_1(1+i) \cup D_1(-1+i)$ and $D_2 = \{z \in \mathbb{C} \mid |\operatorname{Re}(z)| > 1\}$.

9. Find a conformal map that sends D_1 to D_2, where:

   - $D_1 = \{z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1\}$ and $D_2 = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$;
   - $D_1 = \{z \in \mathbb{C} \mid -\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}\}$ and $D_2 = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$;
   - $D_1 = \{z \in \mathbb{C} \mid |z| < 1 \text{ \& } \operatorname{Im}(z) > 0\}$ and $D_2 = D_1(0)$;
   - $D_1 = D_1(0) \setminus (-1, 0]$ and $D_2 = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$;
   - $D_1 = \{z \in C \mid \operatorname{Im}(z) < 0\} \setminus \mathbb{R}_{\leq -1}$ and $D_2 = \{z \in \mathbb{C} \mid -\frac{\pi}{2} < \operatorname{Re}(z) < 0\}$.

10. Let $\Omega = \{z \in \mathbb{C} \mid -1 < \operatorname{Im}(z) < 1\}$ and let f be a holomorphic function in Ω, such that $f(\Omega) \subset \Omega$. Show that if f has two fixed points, i.e. two points such that $f(z) = z$, then f is the identity ($f(z) = z$ for all z).

11. Consider the function $f(z) = \frac{z^2+1}{z}$. Where is f conformal and where it is not? Show that for each $w \in \mathbb{C}$, there exist at most two solutions to the equation $f(z) = w$. Fix $r > 1$, show that the image of the circle $S_r = \{z \in \mathbb{C} \mid |z| = r\}$ is an ellipse and that the image of the circle $S_{1/r} = \{z \in \mathbb{C} \mid |z| = \frac{1}{r}\}$ is the same ellipse. This function is called the Joukowski transform.